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The construction of discontinuous controls for non-linear dynamical system with uncertainties☆

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Abstract

The properties of the solutions of a family of non-linear parametric problems are investigated in the neighbourhood of regular and irregular values of the parameter. Rules are proposed for constructing feedback-type controlling for non-linear dynamical systems with perturbations. An example is presented.

An important factor, which has to be taken into account when constructing control laws for dynamical systems, is uncertainties of different kinds. An uncertainty in the form of a disturbance, which is unknown in advance and acts on the system, is considered. In accordance with the commonly accepted approaches,[1–3] feedback-type controls are constructed which are formulated using the solutions of auxiliary problems that depend on a parameter and are problems for the optimal control of a dynamical system which is non-linear with respect to the state and linear with respect to the control. It is assumed that the optimal control of the auxiliary parametric problem is a bang-bang control. The necessary and sufficient conditions for optimality have already been obtained for problems with bang-bang control,[4,5] an analysis of the sensitivity of the solutions has been carried out and the differentiability of the switching points in the neighbourhood of a regular value of the parameter has been investigated.[6,7]

When parametric problems are used to construct feedback, the parameter changes in a specified interval of the control, and situations when the value of the parameter will not be regular therefore cannot be avoided. On account of this, the properties of the solutions are investigated below in the neighbourhood of both regular as well as irregular values of the parameter. Such problems have not been previously investigated.

The principle of the construction of a feedback which is used below leads to the fact that the control is discontinuous. As a consequence of this, the initial dynamical system, which is looped by such feedback, will be discontinuous with respect to its state and cannot have classical solutions.[8] Most attention is paid to investigating just such situations. Rules for constructing the control in each of the possible situations are described and constructive rules for identifying then are proposed.

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1. Formulation of the problem

Consider the controlled dynamical system

$$dz/dt = f(z,u) + w(t),\quad z = z(t) \in \mathbb{R}^n,\quad t \in T = [0, t_*];\quad f(z,u) = f_0(z) + bu \quad (1.1) \tag{1.1}$$

where z is a state vector, $u = u(t)$ is a scalar control, $f_0(z) \in \mathbb{R}^n$ is a specified fairly smooth function, b is a specified n-vector and $w(t)$, $t \geq 0$ is a disturbance which is not known in advance and acts when the system is functioning. We shall assume that $w(t)$ is any piecewise-continuous n-vector function.

It is required to transfer system (1.1) from a given initial state $x_0 \in \mathbb{R}^n$ at the instant of time $t = 0$ to a specified final state $x_* \in \mathbb{R}^n$ at the instant of time $t = t_*$. The control quality is estimated by the functional

$$\max_{t \in T} |u(t)| \to \min \tag{1.2}$$

Feedback-type controls are customarily used to control systems with uncertainties (interference) Feedback

$$u(\tau, z), (\tau, z) \in T \times \mathcal{G}, \mathcal{G} \subset \mathbb{R}^n$$

is often constructed[1–3] according to the principle

$$u(\tau, z) = u^0(\tau|\tau, z),\quad (\tau, z) \in T \times \mathcal{G} \tag{1.3}$$

where $u^0(t|\tau, z)$, $t \in T_\tau = [\tau, t_*]$ is the solution of the parametric problem

$$\overline{OC}(\tau, z):\begin{cases} J_\tau(u) := \max\limits_{t \in T_\tau} |u(t)| \to \min \\ dx/dt = f(x,u),\quad x(\tau) = z,\quad x(t_*) = x_*,\quad t \in T_\tau \end{cases} \tag{1.4}$$

The concepts of admissible control $u(\cdot|\tau, z) = (u(t|\tau, z), t \in T_\tau)$ and optimal $u^0(\cdot|\tau, z)$ and the trajectories corresponding to them $x(\cdot|\tau, z)$, $x^0(\cdot|\tau, z)$ in the case of problem (1.4) are introduced in the standard way.[9,10] The pair $(x^0(\cdot|\tau, z), u^0(\cdot|\tau, z))$ is called the (strongly) locally optimal solution[11] if a $\varepsilon > 0$ exists such that

$$J_\tau(u(\cdot|\tau, z)) \geq J_\tau(u^0(\cdot|\tau, z))$$

for any admissible pair $x(\cdot|\tau, z)$, $u(\cdot|\tau, z)$, for which the condition

$$\max_{t \in T_\tau} \left\| x(t|\tau, z) - x^0(t|\tau, z) \right\| < \varepsilon$$

is satisfied.

On the basis of known results in Ref 9, it can be shown that an optimal control exists in problem (1.4) if the set of admissible controls is not empty. By virtue of the non-linearity of the problem, cases are possible when several locally-optimal solutions exist. However, in this paper, the following proposition concerning the uniqueness of the solution of problem (1.4) is assumed to be satisfied.

Proposition 1. *For the pair* $(t, z) \in T \times G$, *the problem* $\overline{OC}(\tau, z)$ *has a unique locally optimal solution.*

When the feedback (1.3) is used, the behaviour of the dynamical system (1.1) is described by the equation

$$dz(t)/dt = f(z(t), u(t, z(t))) + w(t), z(0) = x_0 \tag{1.5}$$

in the which the right-hand side, in the general case, is a discontinuous function of the position. It is well known[8] that, in such cases, a classical solution of problem (1.5) may not exist.

The aim of this paper is to construct controls using the feedback (1.3) for dynamical systems with perturbations.

According to relations (1.3)–(1.5), the control $u^*(\tau) = u(\tau, z(\tau))$ at the instant τ in a system which is looped by feedback is constructed using the solution of the parametric problem $OC(\tau) := \overline{OC}(\tau, z(\tau))$, where $z(\tau)$ is the currect state of the actual system, which is assumed to be known at the instant τ. Consequently, to develop effective methods for constructing controls $u^*(\tau)$, $\tau \geq 0$ and to study their properties, it is necessary to investigate the properties of the solutions of the parametric problems $OC(\tau)$, $\tau \geq 0$. These problems are considered in the following section.

2. The parametric optimal control problem

As was mentioned above, in order to construct a feedback it is necessary to investigate the properties of the solutions of the parametric problems $OC(\tau)$, $\tau \geq 0$, which can be written in the form

$$OC(\tau):\begin{cases}\rho \to \min \\ dx/dt = f_0(x) + bu, \quad x(\tau) = z(\tau), \quad x(t_*) = x_* \\ |u(t)| \leq \rho, \quad \rho \geq 0, \quad t \in T_\tau\end{cases} \tag{2.1}$$

Here, $z(\tau)$ is a certain continuous n-vector function, the value of which is assumed to be known when the problem $OC(\tau)$ is considered, and τ is a parameter.

Suppose problem (2.1) is solved for a certain fixed value of the parameter $\tau_0 \geq 0$. It is required to find the solution of the problems $OC(\tau)$, $\tau \in E^+(\tau_0)$ using the known solution of the problem $OC(\tau_0)$. Here, $E^+(\tau_0)$ is the right-side neighbourhood of the point τ_0.

1°. *Definitions. Necessary conditions of optimality*. We will first describe the properties of the solutions of problem (2.1) for a fixed value of the parameter τ under the assumption that a admissible control exists for it.

Suppose $u^0_\tau(\cdot) = (u^0_\tau(t), t \in T_\tau)$, and $x^0_\tau(\cdot) = (x^0_\tau(t), \ \in T_\tau)$ are the optimal control and trajectory of problem $OC(\tau)$. We shall consider the system

$$\Delta\dot{x} = (\partial f_0(x^0_\tau(t))/\partial x)\Delta x + b\Delta u, \quad \Delta x(\tau) = 0 \tag{2.2}$$

Proposition 2. *System (2.2) is T_τ-controllable,*[12] *that is, for any vector $g \in \mathbb{R}^n$, a piecewise-continuous function $\Delta u(t)$, $t \in T_\tau$ exists such that the equality $\Delta x(t_*) = g$ is satisfied on the corresponding trajectory of system (2.2).*

The following theorem can be proved on the basis of Proposition 2 and the classical maximum principle.[10]

Theorem 1. *Suppose $u^0_\tau(\cdot)$ and $x^0_\tau(\cdot)$ are the optimal control and trajectory of problem (2.1) and Proposition 2 is satisfied. A vector $\psi_*(\tau) \in \mathbb{R}^n$ then exists such that, along the solutions $\psi_\tau(t)$, $t \in T_\tau$ of the conjugate system*

$$\dot{\psi}^T = -\psi^T \partial f_0(x^0_\tau(t))/\partial x, \quad \psi(t_*) = \psi_*(\tau) \tag{2.3}$$

the conditions

$$\psi^T_\tau(t)bu^0_\tau(t) = \max_{|u| \leq \rho(\tau)} \psi^T_\tau(t)bu, \quad t \in T_\tau, \quad \rho(\tau) = \max_{t \in T_\tau}|u^0_\tau(t)| \tag{2.4}$$

$$\int_\tau^{t_*}|\psi^T_\tau(t)b|dt = 1, \quad \text{если} \quad \rho(\tau) > 0; \quad \int_\tau^{t_*}|\psi^T_\tau(t)b|dt \leq 1, \quad \text{если} \quad \rho(\tau) = 0 \tag{2.5}$$

are satisfied.

We will denote the set of all vectors $\psi_*(\tau)$ which satisfy relations (2.3)–(2.5) by $Q(\tau)$. By analogy with non-linear programming, we shall call the vector $\psi_*(\tau) \in Q(\tau)$ the Lagrange vector.

Suppose $\psi_*(\tau) \in Q(\tau)$ and $\psi_\tau(t)$, $t \in T_\tau$, are the corresponding solution of the conjugate system. We now construct the switching function

$$\Delta_\tau(\cdot) = (\Delta_\tau(t), t \in T_\tau) = (\psi^T_\tau(t)b, t \in T_\tau) \tag{2.6}$$

Proposition 3. *The function $\Delta_\tau(\cdot)$ vanishes at a finite number of points.*

In this case, according to the maximum principle, the optimal control will be a bang-bang control and it is given by the formula

$$u_\tau^0(t) = \rho(\tau)\operatorname{sign}\Delta_\tau(t), \quad t \in T_\tau$$

Proposition 4. *We shall assume that* $\rho(\tau) \neq 0$ *and consider a certain vector* $\psi_*(\tau) \in Q(\tau)$ *and the corresponding switching function (2.6). We will denote the zeroes of this switching function by* $t_j(\tau)(j = 1, \ldots, p(\tau))$:

$$\{t_j(\tau), j = 1, \ldots, p(\tau)\} = \{t \in T_\tau: \Delta_\tau(t) = 0\}, \quad t_{j-1}(\tau) < t_j(\tau), \quad j = 2, \ldots, p(\tau)$$

We will denote the multiplicity of a zero $t_j(\tau)$ *of the function* $\Delta_\tau(\cdot)$ *by* $\nu_j = \nu_j(\tau)$:

$$\partial^\nu \Delta_\tau(t)/\partial t^\nu\big|_{t = t_j(\tau)} = 0 \text{ при } \nu = 0, \ldots, \nu_j; \quad \partial^{\nu_j+1}\Delta_\tau(t)/\partial t^{\nu_j+1}\big|_{t = t_j(\tau)} \neq 0$$

The following assertion can be proved using the results obtained earlier in Ref 13.

Assertion 1. Suppose Propositions 2 and 4 are satisfied. The set $Q(\tau)$ is then bounded and a vector $\psi_*(\tau) \in Q(\tau)$ is found such that the condition

$$\operatorname{rank}(q_\tau^{(\nu)}(t_j(\tau)), \nu = 0, \ldots, \nu_j, j = 1, \ldots, p(\tau), q_\tau^*) = n \tag{2.7}$$

is satisfied.

Here,

$$q_\tau(t) := F_\tau(t_*, t)b, \quad q_\tau^{(\nu)}(t) = \frac{\partial^\nu q_\tau(t)}{\partial t^\nu}, \quad q_\tau^* = \int_{T_\tau} q_\tau(t)\operatorname{sgn}(\Delta_\tau(t))dt$$

and $F_\tau(t, \upsilon)$ is the fundamental matrix of the solutions of the system $\Delta\dot{x} = (\partial f_0(x_\tau^0(t))/\partial x)\Delta x$.

We shall call the vector $\psi_*(\tau) \in Q(\tau)$ which satisfies conditions (2.7) a basis vector. Without loss of generality, we shall always subsequently assume that the vector $\psi_*(\tau) \in Q(\tau)$ being considered is a basis vector.

Suppose $u_\tau^0(\cdot)$ and $x_\tau^0(\cdot)$ are the optimal control and trajectory of the problem $OC(\tau)$, $\psi_*(\tau) \in Q(\tau)$ is the Lagrange vector and $\Delta_\tau(\cdot)$ is the switching function corresponding to them. We now introduce the sets of parameters

$$S(\tau) = \{p(\tau), k(\tau), l_*(\tau), l^*(\tau), L(\tau)\}, \quad \Theta(\tau) = (t_j(\tau), j = 1, \ldots, p(\tau); \rho(\tau); \psi_*(\tau))$$

Here,

$$l_*(\tau) = 1, \text{ если } t_1(\tau) = \tau; \; l_*(\tau) = 0, \text{ если } t_1(\tau) > \tau$$
$$l^*(\tau) = 1, \text{ если } t_{p(\tau)}(\tau) = t_*; \; l^*(\tau) = 0, \text{ если } t_{p(\tau)}(\tau) < t_*$$
$$k(\tau) = \operatorname{sgn} u_\tau^0(\tau + 0), \; L(\tau) = \{j \in \{1, \ldots, p(\tau)\}: \nu_j(\tau) > 0\}$$

Definition 1. We will call the sets $S(\tau)$ and $\Theta(\tau)$ the structure and the defining elements of the solution of the problem $OC(\tau)$.

Note that these sets $S(\tau)$ and $\Theta(\tau)$ are a finite set of data. They enable as to recover the optimal bang-bang control $u_\tau^0(\cdot)$ of the problem $OC(\tau)$ in a unique manner and to check the conditions of the maximum principle.

The following definitions are necessary in order to introduce the concept of regularity.

We put

$$p = p(\tau), \quad k = k(\tau), \quad \nu_j = \nu_j(\tau), \quad j = 1, \ldots, p, \quad m_* = p + \sum_{j=1}^{p} \nu_j$$

and introduce the vector of the parameters

$$\xi = (\mu_j, j = 1, \ldots, m_*; \rho)$$

and the function

$$\mathcal{L}(\bar{\lambda}, \xi, \tau) = \lambda_0\rho + \lambda^T x(t_*|\xi, \tau); \quad \bar{\lambda} = (\lambda_0, \lambda), \quad \lambda_0 \in \mathbb{R}, \quad \lambda \in \mathbb{R}^n$$

into consideration, where $x(t|\xi, \tau)$, $t \in T_\tau$ is the continuous solution of the system of differential equations

$$\dot{x} = f_0(x) + b(-1)^j k\rho, \quad t \in [\mu_{j-1}, \mu_j), \quad j = 1, \ldots, m_*$$
$$x(\tau) = z(\tau), \quad \mu_0 = \tau, \quad \mu_{m_*+1} = t_*$$

Using the defining elements of the optimal bang-bang control $u^0_\tau(\cdot)$ of problem (2.1), we form the vector $\xi(\tau) = (\mu_j(\tau), j = 1, \ldots, m_*; \rho(\tau))$:

$$\{\mu_j(\tau), j = 1, \ldots, m_*\} = \{t_{\nu j} := t_j(\tau), \nu = 0, \ldots, \nu_j, j = 1, \ldots, p\}$$
$$\mu_j(\tau) \le \mu_{j+1}(\tau), \quad j = 1, \ldots, m_* - 1$$

It has been proved* that the set

$$\bar{\Lambda} = \{\bar{\lambda} \in \mathbb{R}^{n+1}: \partial\mathcal{L}(\bar{\lambda}, \xi(\tau), \tau)/\partial\xi = 0, \bar{\lambda} \ne 0\}$$

is non-empty and, in particular, that $(1, -\psi_*(\tau)) \in \bar{\Lambda}$ and

$$\max_{\bar{\lambda} \in \bar{\Lambda}} l^T(\partial^2\mathcal{L}(\bar{\lambda}, \xi(\tau), \tau)/\partial\xi^2)l \ge 0, \ \forall l \in K_\tau(\xi(\tau))$$

where

$$K_\tau(\xi) = \{l = (l_j, j = 1, \ldots, m_*; l_\rho) : l^T\partial x(t_*|\xi, \tau)/\partial\xi = 0, l_{j-1} \le l_j,$$
$$\text{если } \mu_{j-1} = \mu_j, j = 1, \ldots, m_* + 1\}, \quad \mu_0 = l_0 = \tau, \quad \mu_{m_*+1} = l_{m_*+1} = t_*$$

Definition 2. The optimal bang-bang control $u^0_\tau(\cdot)$ is said to be regular and the current value of the parameter τ is a regular point if

$$\beta(\tau) := l_*(\tau) + l^*(\tau) + \sum_{i=1}^{p(\tau)} \nu_i(\tau) = 0$$

and the condition

$$l^T(\partial^2\mathcal{L}((1, -\psi_*(\tau)), \xi(\tau), \tau)/\partial\xi^2)l > 0, \quad \forall l \in K_\tau(\xi(\tau))\backslash\{0\} \tag{2.8}$$

is satisfied.

By analogy with what has been stated earlier in Ref 13, it can be proved that the regularity (non-regularity) of a point τ is independent of the choice of the basis vector $\psi_*(\tau) \in Q(\tau)$. We note that, in the case of a regular optimal control $u^0_\tau(\cdot)$, the set $\bar{\Lambda}$ consists of the unique vector $\hat{\lambda} = (1, -\psi_*(\tau))$ and the relations $n \le p(\tau) + 1$, $p(\tau) = m(\tau)$ hold, where $m(\tau)$ is the number of switching points of the optimal control $u^0_\tau(\cdot)$. The rules for constructing the matrices $\partial x(t_*|\xi(\tau), \tau)/\partial\xi$, $\partial^2 I(\hat{\lambda}, \xi(\tau), \tau)/\partial\xi^2$ were presented earlier in Ref. 14.

* Kostyukova OI, Kurdina MA, The control of non-linear dynamical systems with perturbations using the solutions of parametric optimal control problems. Preprint No. 2 (573) Inst. Mat., Narod. Akad. Nauk Belarusi, Minsk, 2005.

2°. *Construction of the solutions of the family of problems $OC(\tau)$ in the neighbourhood of a regular point*. Suppose that, for a certain value of the parameter $\tau=\tau_0$, the problem $OC(\tau_0)$ has an optimal regular control $u^0_\tau(\cdot)$ with a structure and defining elements

$$S(\tau_0) = \{p(\tau_0), k(\tau_0), l_*(\tau_0) = 0, l^*(\tau_0) = 0, L(\tau_0) = \varnothing\} \tag{2.9}$$

$$\Theta^0 = \Theta(\tau_0) = (t^0_j = t_j(\tau_0), j = 1, \dots, p(\tau_0); \rho^0 = \rho(\tau_0); \psi^0_* = \psi_*(\tau_0)) \tag{2.10}$$

It is required to determine simple rules for constructing the solutions $u^0_\tau(\cdot)$ of problems $OC(\tau)$ when $\tau \in E^+(\tau_0)$. In the case of a regular value of the parameter τ_0, these rules are specified by the theorem presented below which was proved earlier in Ref. 14. We now introduce the necessary notation for formulating of this theorem.

Suppose the numbers p and k are specified: $p \in \mathbb{N}$, $k = \pm 1$. We consider the $(p+1+n)$-vector

$$\Theta = (t_j, j = 1, \dots, p; \rho; \psi_*), \quad \psi_* \in \mathbb{R}^n \tag{2.11}$$

and denote the solution of the system of equations

$$\dot{x} = f_0(x) + bu(t|\Theta), \quad x(\tau) = z(\tau) \tag{2.12}$$

by $x(t|\Theta, \tau)$, $t \in T_\tau$, where

$$u(t|\Theta) = k(-1)^j\rho, \quad t \in [t_j, t_{j+1}), \quad j = 0, \dots, p; \quad t_0 = \tau, \quad t_{p+1} = t_*$$

and the solution of the conjugate system

$$\dot{\psi}^T(t) = -\psi^T(t)\partial f^T_0(x(t|\Theta, \tau))/\partial x, \quad \psi(t_*) = \psi_* \tag{2.13}$$

by $\psi(t|\Theta, \tau)$, $t \in T_\tau$.

We now consider the $(n+p+1)$-vector function

$$\Phi(\Theta, \tau) := \left\| \begin{matrix} x(t_*|\Theta, \tau) - x_* \\ 2(-1)^{j-1}k\Delta(t_j|\Theta, \tau), \ j = 1, \dots, p \\ \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} (-1)^j k\Delta(t|\Theta, \tau)dt - 1 \end{matrix} \right\|; \ \Delta(t|\Theta, \tau) = \psi^T(t|\Theta, \tau)b, \ t \in T_\tau \ (2.14) \tag{2.14}$$

We emphasize that the form of the vector Θ and the vector function $\Phi(\Theta, \tau)$, as well as their dimensions, depend on the parameters p and k, which are assumed to be specified.

Theorem 2. *Suppose the problem $OC(\tau_0)$ has an optimal regular control $u^0_\tau(\cdot)$ with a structure (2.9) and defining elements (2.10). Then, when $k=k(\tau_0)$, $p=p(\tau_0)$,*

1) *a unique continuous $(p+n+1)$-vector function*

$$\Theta(\tau) = (t_j(\tau), j = 1, \dots, p; \rho(\tau); \psi_*(\tau)), \quad \tau \in \mathscr{E}^+(\tau_0) \tag{2.15}$$

exists, which satisfies the relations

$$\Phi(\Theta(\tau), \tau) \equiv 0, \quad \tau \in \mathscr{E}^+(\tau_0); \quad \Theta(\tau_0) = \Theta^0 \tag{2.16}$$

2) *for $\tau \in E^+(\tau_0)$, the control $u^0_\tau(\cdot)$:*

$$u^0_\tau(t) = k(-1)^j\rho(\tau), \ t \in [t_j(\tau), t_{j+1}(\tau)), \ j = 0, \dots, p; \ t_0(\tau) = \tau, \ t_{p+1}(\tau) = t_* \ (2.17) \tag{2.17}$$

has the constant structure $S(\tau) \equiv S(\tau_0)$ and is optimal in the problem $OC(\tau)$.

It follows from Theorem 2 that if the solution of the unperturbed problem $OC(\tau_0)$, which corresponds to the regular value of the parameter τ_0, is known, then the construction of the solutions $u^0_\tau(\cdot)$ of the perturbed problems $OC(\tau)$, $\tau \in E^+(\tau_0)$ reduces to finding the solutions $\Theta(\tau)$ of the systems of non-linear equations (2.16). The optimal control $u^0_\tau(\cdot)$ is constructed according to relations (2.17) using the elements of the vector $\Theta(\tau)$.

3. Construction of the feedback

1°. *Construction of the feedback in the neighbourhood of a regular point*. We will consider a certain control of system (1.1). Suppose τ_0 is the current instant in real time and $z(\tau_0)$ is the known current state of the system. We will assume that the solution, which is regular, that is, τ_0 is a regular point, is known for the problem $OC(\tau_0) = \overline{OC}(\tau_0, z(\tau_0))$.

Remark 1. We emphasize that it is impossible to determine in advance (up to the advent of the instant of time τ_0) whether the point τ_0 will be regular or not, since, in each real process, the regularity or irregularity of the point τ_0 depends on the state of the system $z(\tau_0)$ generated by the perturbation $w(t)$, $t \in [0, \tau_0)]$ which has been realized up to this instant.

For $\tau \in E^+(\tau_0)$, we shall construct the feedback according to the rule (1.3) which, when account has been taken of the results in Section 2, we can write in the form

$$u^*(\tau) = u(\tau, z(\tau)) = u^0_\tau(\tau) = k\rho(\tau) \tag{3.1}$$

Here, $\rho(\tau)$ is the component of the vector function $\Theta(\tau)$ (see (2.15) which is the solution of system (2.16). This system is uniquely formed when $k = k(\tau_0)$ and $p = p(\tau_0)$ using the solutions of systems (2.12) and (2.13) in which $z(\tau) = z(\tau|u^*_\tau(\cdot), w_\tau(\cdot))$ is the current state (at the instant τ) of the real system

$$\dot{z}(t) = f(z(t), u^*(t)) + w(t), \quad t \in [\tau_0, \tau); \quad z(\tau_0 + 0) = z(\tau_0) \tag{3.2}$$

where $w_\tau(\cdot) = w(t)$, $t \in [0, \tau))$ is the perturbation acting on the system during its operation and $u^*_\tau(\cdot) = (u^*(t), t \in [0, \tau))$ is the control constructed up to the current instant τ.

The feedback is constructed according to the rule (3.1) in the interval $\tau \in [\tau_0, \tau_1]$, where τ_1 is the instant when a) $\beta(\tau_1) \neq 0$ or b) $\tau_1 = t_* - \varepsilon$. Here, $\gamma > 0$, a fairly small number, is the parameter of the algorithm. In case *a*, the regularity conditions are violated in the case of the problem $OC(\tau_1)$ and the feedback is subsequently constructed according to the rules described in the following subsection. In case *b*, the construction of the feedback is finished. The need to stop the construction of the feedback was due to the fact that, when account is taken of the assumptions in this paper, irregularity arises ever more frequently as the real instant of time τ approaches t_*. It is therefore reasonable to stop the process of feedback construction at the instant $\tau_1 = t_* - \varepsilon$ and, when $\tau \in [t_* - \varepsilon, t_*]$ to feed a control $u^*(\tau) = u^0_\tau, (\tau)$, which is the solution of the problem $OC(\tau_0)$, into the input of the system.

2°. *Construction of the feedback in the neighbourhood of an irregular point*. As previously, suppose the current state $z(\tau_0)$ of the actual system and the solution $u^0_\tau(\cdot)$ of the problem $OC(\tau_0) = \overline{OC}(\tau_0, z(\tau_0))$ are known for the current instant τ_0. We will now assume that the value of τ_0 is irregular.

We shall consider the case $\beta(\tau_0) = 1$. Condition (2.8) is assumed to be satisfied. In this case, irregularity can arise for one of the following reasons.

Situation 1: $l_*(\tau_0) = 1$. This means that the first zero of the switching function $\Delta_{\tau_0}(\cdot)$ coincides with the current instant τ_0: $t_1(\tau_0) = \tau_0$. The corresponding switching function is represented by the thin curve in Fig. 1.

Situation 2: $l^*(\tau_0) = 1$. This means that the last zero of the switching function $\Delta_{\tau_0}(\cdot)$ coincides with the instant $t_* : t_{p(\tau_0)}(\tau_0) = t_*$. The corresponding switching function is represented by the dashed curve line in Fig. 2.

Situation 3: $\sum_{j=1}^{p(\tau_0)} \nu_j(\tau_0) = 1$. This means that the switching function $\Delta_{\tau_0}(\cdot)$ has a multiple zero. The corresponding switching function is represented by the dashed curve in Fig. 3.

Unlike a regular point, an irregular point τ_0 is characterized by the fact that, in the general case, there is a change in the structure of the solution of the problems $OC(\tau)$: $S(\tau_0) \neq S(\tau_0 + 0)$ at this point.[15] The new structure $S(\tau_0 + 0)$ belongs to a certain set $P(\tau_0)$ of structures which are permitted at the point τ_0. It will be shown later that the actual

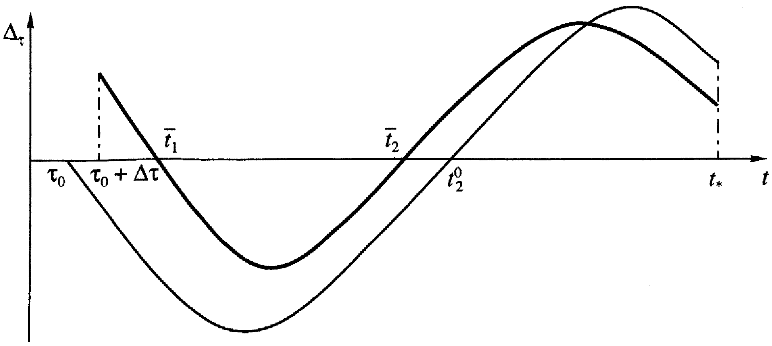

Fig. 1.

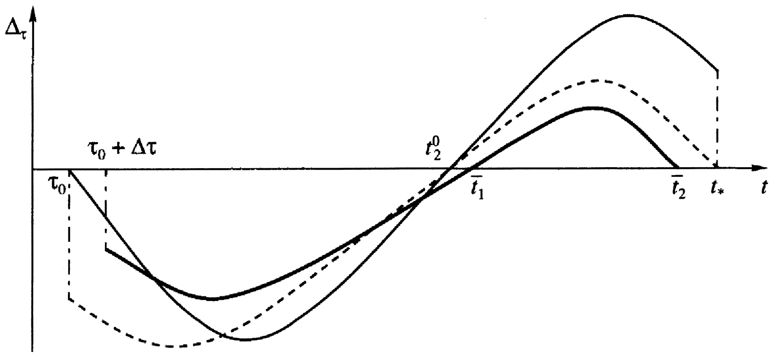

Fig. 2.

form of the new structure $S(\tau_0+0) \in P(\tau_0)$ can be determined knowing $w(\tau_0+0)$, and, in considering the current irregular point τ_0, we shall therefore henceforth assume that the current state $z(\tau_0)$ and the perturbation $w(\tau_0+0)$ are known.

We will investigate the degenerate Situations 1–3 under the assumption that $n = p(\tau_0)+1$ and, consequently, $n = m(\tau_0)+2$.

Situation 1. Since the feedback (1.3) is constructed using the solution of the parametric problems $OC(\tau)$, we will investigate the properties of these problems in the neighbourhood $E_+(\tau_0)$ of the irregular point τ_0. Suppose the optimal bang-bang control $u^0_{\tau_0}(\cdot)$ of the problem $OC(\tau_0)$ has the following structure and defining elements

$$S(\tau_0) = \{p(\tau_0), k(\tau_0), l_*(\tau_0) = 1, l^*(\tau_0) = 0, L(\tau_0) = \varnothing\} \tag{3.3}$$

$$\Theta^0 = \Theta(\tau_0) = (t_j^0, j = 1, \ldots, p(\tau_0); \rho(\tau_0); \psi_*(\tau_0)) \tag{3.4}$$

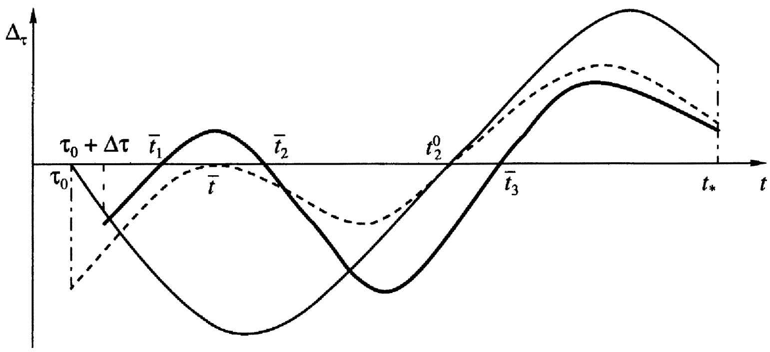

Fig. 3.

The structure of the solution is determined by the switching function, and the behaviour of this function when $\tau \in E^+(\tau_0)$ changes is therefore primarily of interest.

When $\tau \in E^+(\tau_0)$, the following cases are possible in problems $OC(\tau) = \overline{OC}(\tau, z(\tau))$.

Case A. The Lagrange function $\psi_*(\tau)$, $\tau \in E^+(\tau_0)$ is continuous at the point

$$\tau_0 + 0\colon\ \psi_*(\tau_0) = \psi_*(\tau_0 + 0)$$

The switching function $\Delta_\tau(\cdot)$ will have the form shown by the thick curve in Fig. 1.

Case B. The Lagrange function $\psi_*(\tau)$, $\tau \in E^+(\tau_0)$ is discontinuous at the point

$$\tau_0 + 0\colon\ \psi_*(\tau_0) \neq \psi_*(\tau_0 + 0)$$

The new function $\Delta_{\tau_0+0}(\cdot)$, which corresponds to the new vector $\psi_*(\tau_0 + 0)$, will have the form shown by the dashed curve in Fig. 2 (we call this subcase $\tilde{A}$) or in Fig. 3 (we call this subcase $\tilde{\text{Б}}$). When $\tau \in E^+(\tau_0)$, the switching function $\Delta_\tau(\cdot)$ will have the form shown by the heavy curve in Fig. 2 or Fig. 3.

Case C. The Lagrange function $\psi_*(\tau)$, $\tau \in E^+(\tau_0)$ is continuous at the point

$$\tau_0 + 0\colon\ \psi_*(\tau_0) = \psi_*(\tau_0 + 0)$$

The identity $t_1(\tau) \equiv \tau$, $\tau \in E^+(\tau_0)$ is satisfied for the switching function $\Delta_\tau(\cdot)$.

We will now show how, knowing the value of just a single parameter (λ_1), which is calculated on the basis of the data available at the instant τ_0, it is possible to determine which of the Cases A–C will be realized for $\tau \in E^+(\tau_0)$.

We put

$$k := -k(\tau_0) = -\text{sgn}\, u^0_{\tau_0}(\tau_0 + 0), \quad p := p(\tau_0) = n - 1 \tag{3.5}$$

and introduce the vector of the parameters Θ given by formula (2.11). The solutions of the direct and conjugate systems (2.12) and (2.13) are denoted by $x(t|\Theta, \tau)$ and $\psi(t|\Theta, \tau)$, $t \in T_\tau$ respectively and the function $\Phi(\Theta, \tau)$ is determined according to the rules (2.14), using parameters k and p, specified by relations (3.5).

It was shown in Ref. 14 that

$$\det \partial\Phi(\Theta^0, \tau_0)/\partial\Theta \neq 0, \quad \Phi(\Theta^0, \tau_0) = 0$$

Using these relations and the implicit function theorem, we conclude that a unique continuous vector function $\Theta(\tau)$ exists (see (2.15)) which satisfies relations (2.16).

We obtain

$$\dot{t}_1(\tau_0) = -e_1^T(\partial\Phi(\Theta^0, \tau_0)/\partial\Theta)^{-1}\partial\Phi(\Theta^0, \tau_0 + 0)/\partial\tau =: \gamma_1 \tag{3.6}$$

We shall distinguish between the cases $\gamma_1 > 1$, $\gamma_1 < 0$, $\gamma_1 \in (0, 1)$ consider each of them and indicate their relation to Cases A–C described above. The cases $\gamma_1 = 0$ and $\gamma_1 = 1$, which require additional investigation, are not considered here.

The case $\gamma_1 > 1$. It can be shown (see the footnote in Section 2) that, if $\gamma_1 > 1$, the relation $t_1(\tau) > \tau$ is satisfied for $\tau \in E^+(\tau_0)\backslash\tau_0$ which means that the first zero $t_1(\tau)$ of the switching function $\Delta_\tau(\cdot)$ departs to the right, forming a new switching point $t_1(\tau) > \tau$ of the control $u^0_{\tau_0}(\cdot)$. In other words, when $\tau \in E^+(\tau_0)\backslash\tau_0$, the switching function $\Delta_\tau(\cdot)$ will have the form represented by the heavy curve in Fig. 1 and the control, which is constructed according to the rules (2.17) using the solution $\Theta(\tau)$ of system (2.16) in which the parameters Θ^0, k and p are determined from (3.4) and (3.5), has a constant structure of the form

$$S(\tau) = S(\tau_0 + 0) = \{p(\tau_0), -k(\tau_0), 0, 0, \varnothing\}$$

and is the optimal control in the problem $OC(\tau)$, $\tau \in E^+(\tau_0)$.

It follows from the arguments which have been presented that Case A described above will hold when the inequality $\gamma_1 > 1$ is satisfied. For $\tau \in E^+(\tau_0)$, the feedback (1.3) is constructed in the same way as in the regular case: according to the rules (3.1), where $\rho(\tau)$ is the component of the vector $\Theta(\tau)$ which is the solution of system (2.16) in which the parameters Θ^0, k and p are defined by relations (3.4) and (3.5), and $z(t)$ is the current state of system (3.2).

The case $\gamma_1 < 0$. Since $\gamma_1 < 0$, the inequality

$$t_1(\tau) < \tau, \quad \tau \in \mathscr{E}^+(\tau_0)\backslash\tau_0$$

is true for the components $t_1(\tau)$ of the solution $\Theta(\tau)$ of the system of non-linear equations (2.16), where Θ^0, k and p are given by relations (3.4) and (3.5).

Consequently, this vector $\Theta(\tau)$ cannot be used to construct a control using the rules (3.1) since a control constructed using these rules will not be an admissible control.

In the Situation 1 being considered, the set $Q(\tau_0)$ of Lagrange vectors consists of more than a single vector. We construct (see footnote in Section 2) a new Lagrange vector

$$\bar{\psi}_* := \psi_*(\tau_0 + 0) \in Q(\tau_0)$$

and denote the new switching function corresponding to the vector $\bar{\psi}_*$ by $\bar{\Delta}_{\tau_0}(\cdot)$. It can have the form shown by the dashed curve in Fig. 2 (*Subcase* Ã) or in Fig. 3 (*Subcase* $\tilde{\text{Б}}$).

We will now consider Subcase Ã. Here, the point $t = \tau_0$ ceases to be a zero of the new switching function and the point $t = t_*$ becomes a (simple) zero of the new switching function. It can be shown (see footnote in Section 2) that, if $\gamma_1 < 0$, then $\bar{\Delta}_{\tau_0}(\cdot) = \Delta_{\tau_0+0}(\cdot)$ and, when $\tau \in E^+(\tau_0)\backslash\tau_0$, the last zero $t_p(\tau)$, $t_p(\tau_0 + 0) = t_*$, of the switching function $\Delta_\tau(\cdot)$ moves a little to the left ($\gamma_p := i_p(\tau_0 + 0) < 0$), forming a new switching point $t_p(\tau) < t_*$ of the control $u^0_\tau(\cdot)$. In other words, when $\tau \in E^+(\tau_0)\backslash\tau_0$, the switching function $\Delta_\tau(\cdot)$ will have the form represent by the heavy curve in Fig. 2 and the control $u^0_\tau(\cdot)$, constructed according to the rules (2.17) where $\Theta(\tau) = (t_j(\tau), j = 1, \ldots, p; \rho(\tau); \psi_*(\tau))$ is the solution of system (2.16) when

$$k := k(\tau_0) = \operatorname{sgn} u^0_{\tau_0}(\tau_0 + 0), \quad p := p(\tau_0) = n - 1 \tag{3.7}$$

with the initial condition

$$\Theta(\tau_0 + 0) = \tilde{\Theta} := (\tilde{t}_j = t^0_{j+1}, j = 1, \ldots, p - 1, \tilde{t}_p = t_*; \rho(\tau_0); \bar{\psi}_*)$$

has a constant structure of the form

$$S(\tau) = S(\tau_0 + 0) = \{p(\tau_0), k(\tau_0), 0, 0, \varnothing\}$$

and is the optimal control in the problem $OC(\tau)$.

We will now consider the subcase $\tilde{\text{Б}}$. Here, the point $t = \tau_0$ again ceases to be a zero of the new switching function and a multiple zero of the new switching function appears at the point $t = \bar{t}$, $\bar{t} \in (t^0_{j0}, t^0_{j_0+1})$, $j_0 \in \{0, 1, \ldots, p(\tau_0)\}$. It can be shown (see footnote in Section 2) that, if $\gamma_1 < 0$, then $\bar{\Delta}_{\tau_0}(\cdot) = \Delta_{\tau_0+0}(\cdot)$ and, when $\tau \in E^+(\tau_0)\backslash\tau_0$, the multiple zero $\bar{t}$ of the new switching function $\bar{\Delta}_{\tau_0}(\cdot)$, when $\tau \in E^+(\tau_0)\backslash\tau_0$, generates two new switching points $t_{j_0}(\tau)$ and $t_{j_0+1}(\tau)$, $t_{j_0}(\tau_0 + 0) = t_{j_0+1}(\tau_0 + 0) = t^-$, of the optimal control of problem $OC(\tau)$ for which

$$k := k(\tau_0) = \operatorname{sign} u^0_{\tau_0}(\tau_0 + 0), \quad p := p(\tau_0) + 1 = n$$

This means that, when $\tau \in E^+(\tau_0)\backslash\tau_0$, the switching function $\Delta_\tau(\cdot)$ will have the form represented by the heavy curve in Fig. 3 and the control $u^0_\tau(\cdot)$, constructed according to the rules (2.17), where $\Theta(\tau)$ is the solution of system (2.16) when

$$\begin{aligned} \Theta(\tau_0 + 0) = \bar{\Theta} := (&\bar{t}_{j-1} = t^0_j, j = 2, \ldots, j_0; \bar{t}_{j_0} = \bar{t}_{j_0+1} = \bar{t}, \bar{t}_{j+1} = t^0_j, \\ &j = j_0 + 1, \ldots, p(\tau_0); \rho(\tau_0); \bar{\psi}_*) \end{aligned} \tag{3.8}$$

with the initial condition

$$\Theta(\tau_0+0) = \bar{\Theta} := (\bar{t}_{j-1} = t_j^0,\ j = 2, \dots, j_0;\ \bar{t}_{j_0} = \bar{t}_{j_0+1} = \bar{t},\ \bar{t}_{j+1} = t_j^0,$$
$$j = j_0+1, \dots, p(\tau_0);\ \rho(\tau_0);\ \bar{\psi}_*)$$

has a constant structure of the form

$$S(\tau) = S(\tau_0+0) = \{p(\tau_0)+1, k(\tau_0), 0, 0, \varnothing\}$$

and is the optimal control in the problem $OC(\tau)$.

It follows from the results obtained that Case B described above holds when the inequality $\gamma_1 < 0$ is satisfied. When $\tau \in E^+(\tau_0)\backslash\tau_0$, the feedback (1.3) is constructed in the same way as in the regular case: using rules (3.1), where $\rho(\tau)$ is the component of the vector $\Theta(\tau)$ (see (2.15)) which is the solution of the system of equations (2.16) in which, in the Subcase Ã, the parameters k and p are determined from relations (3.7) and the initial condition has the form $\Theta(\tau_0+0) = \tilde{\Theta}$ while, in Subcase $\tilde{\text{Б}}$, p and k are determined from relations (3.8) and the initial condition has the form $\Theta(\tau_0+0) = \bar{\Theta}$; $z(\tau)$ is the current state of system (3.2).

The case $\gamma_1 \in (0, 1)$. In this case, the control

$$u^*(\tau) = u(\tau, z(\tau)), \quad \tau \in \mathscr{E}^+(\tau_0)$$

cannot be constructed using any of the rules described in the cases when $\gamma_1 > 1$ and $\gamma_1 < 0$. The case being considered, $t_1(\tau_0) = \tau_0$ and $\gamma_1 \in (0, 1)$, corresponds to a situation when the point $(\tau_0, z(\tau_0))$ is located on the surface of discontinuity J of the function $\bar{f}(z, \tau, w) := f(z, u(\tau, z)) + w$ and, when $\tau \in E^+(\tau_0)$, a classical solution does not exist in system (1.5). According to the approach in Ref. 8, the solution $z(t)$, $t \geq \tau_0$ of system (1.5) will therefore be determined in such a way that the points $(t, z(t))$, when $\tau \in E^+(\tau_0)$, remain on the surface of discontinuity J. It is customary[8] to call this solution a sliding mode.

According to the agreement, we shall construct the control

$$u^*(\tau), \quad \tau \in \mathscr{E}^+(\tau_0) \tag{3.9}$$

in the sliding segments when $\tau \in E^+(\tau_0)$ in such a way that the trajectory of the real system (1.5) slides along the surface of discontinuity J. In terms of the solutions of the problems

$$OC(\tau) := \overline{OC}(\tau, z(\tau|u_\tau^*(\cdot), w_\tau(\cdot))), \quad \tau \in \mathscr{E}^+(\tau_0) \tag{3.10}$$

this condition means that the first zero $t_1(\tau)$ of the switching function of the problem $OC(\tau)$ coincides with the current instant τ:

$$t_1(\tau) \equiv \tau, \quad \tau \in \mathscr{E}^+(\tau_0). \tag{3.11}$$

Here $z(\tau) = z(\tau|u_\tau^*(\cdot), w\tau(\cdot))$, $\tau \in E^+(\tau_0)$ is the state of system (3.2), which is generated by the control $u_\tau^*(\cdot) = (u^*(t), t \in [0, \tau))$ constructed and the perturbation $w_\tau(\cdot) = (w(t), t \in [0, \tau))$ which has been realized.

We will now obtain the relations which enable us to construct control (3.9) which ensures that condition (3.11) is satisfied. Here, we shall assume that the state $z(\tau)$ of the real system and the perturbation $w(\tau+0)$ are known at each current instant τ.

We put

$$k := k(\tau_0) = \text{sgn} u_{\tau_0}^0(\tau_0+0), \quad p := p(\tau_0) = n-1$$

We now consider a $(p+n)$-vector of the parameters $\hat{\Theta} = (t_j,\ j = 2, \dots, p;\ \rho;\ \psi_*)$ and denote the solutions of the direct and conjugate systems

$$\dot{x}(t) = f(x(t), (-1)^j k\rho), \quad \dot{\psi}^T(t) = -\psi^T(t)\partial f_0(x(t))/\partial x$$
$$t \in [t_{j-1}, t_j), \quad j = 2, \dots, p+1, \quad t_1 \equiv \tau$$

by $\hat{x}(t|z, \hat{\Theta}, \tau)$, $\hat{\psi}(t|z, \hat{\Theta}, \tau)$, $t \in T_\tau$ with the boundary conditions

$$x(\tau) = z, \quad \psi(t_*) = \psi_*$$

We will now construct a $(p+n+1)$-vector function $\hat{\Phi}(z, \hat{\Theta}, \tau)$ according to the rules (2.14) in which the functions $x(t_*|\Theta, \tau)$, $\Delta(t|\Theta, \tau)$ are replaced by $\hat{x}(t_*|z, \hat{\Theta}, \tau)$, $\hat{\Delta}(t|z, \hat{\Theta}, \tau) := \hat{\psi}^T(t|z, \hat{\Theta}, \tau)b$ and $t_1 = \tau$.

It has been shown (see footnote in Section 2) that a number $|\tilde{\gamma}| < \rho(\tau_0)$ and unique continuous functions

$$u^*(\tau), \quad \hat{\Theta}(\tau) = (t_j(\tau), j = 2, \ldots, p(\tau); \rho(\tau); \psi_*(\tau)), \quad \tau \in \mathscr{E}^+(\tau_0) \tag{3.12}$$

exists which satisfy the relations

$$\hat{\Phi}(z(\tau|u_\tau^*(\cdot), w_\tau(\cdot)), \hat{\Theta}(\tau), \tau) \equiv 0, \quad \tau \in \mathscr{E}^+(\tau_0)\backslash\tau_0$$

$$u^*(\tau_0+0) = \tilde{\gamma}, \ t_j(\tau_0+0) = t_j^0, \ j = 2, \ldots, p; \ \rho(\tau_0+0) = \rho(\tau_0); \ \psi_*(\tau_0+0) = \psi_*(\tau_0)$$

and that, when $\tau \in E^+(\tau_0)$, the first zero $t_1(\tau)$ of the switching function of problem $OC(\tau)$ (3.10) coincides with the current instant τ; the control $u_\tau^0(\cdot)$, constructed according to the rules (2.17), taking account of identity (3.11), is the optimal control in the problem $OC(\tau)$ and has a constant structure $S(\tau) = S(\tau_0)$ of the form (3.3).

Rules for calculating the number $\tilde{\gamma}$ and for constructing the function (3.12) have also been described (see footnote in Section 2). According to these rules, in order to construct a control $u^*(\tau)$ at the current moment in time τ it is sufficient to know the current state of the system $z(\tau)$ and the current perturbation $w(\tau+0)$. Hence, we have feedback with respect to the current state and perturbation in the intervals where there is sliding.

Taking account of the relation

$$|\tilde{\gamma}| < \rho(\tau_0+0), \quad u^*(\tau_0+0) = \tilde{\gamma}$$

and the continuity of the functions $u^*(\tau)$, $\rho(\tau)$, $\tau \in E^+(\tau_0)\backslash\tau_0$, we conclude that the following inequality holds

$$|u^*(\tau)| < \rho(\tau), \quad \tau \in \mathscr{E}^+(\tau_0)\backslash\tau_0$$

It follows from the results obtained that, when $\gamma_1 \in (0, 1)$, Case C described above will hold.

The control $u^*(\tau)$, constructed using the proposed rules, is fed into the input of system (3.2) in the interval $\tau \in (\tau_0, \tau_*)$, where $\tau_1 > \tau_0$ is the instant which is the closest from the right to τ_0 for which the equality $\tau_1 = t_*$ or $u^*(\tau_1)$ or $\beta(\tau_1) > 1$ is satisfied.

The first equality means that the process of constructing the feedback has been completed. The second equality means that, when $\tau > \tau_1$, the trajectory of the real system leaves the surface of discontinuity and, in order to construct the control $u^*(\tau)$, $\tau > \tau_1$, the rules described above for the cases when $\gamma_1 > 1$ and $\gamma_1 > 0$ are used, depending on which of the equalities $u^*(\tau_1) = \rho(\tau_1)$ or $u^*(\tau_1) = -\rho(\tau_1)$ holds. The cases $\beta(\tau_1) > 1$ are the subject of an independent investigation and are not considered in this paper.

Situation 2. This situation is investigated in an analogous manner to Situation 1 only now all the constructions start out from the switching function $\Delta_{\tau_0}(\cdot)$ represented by the dashed curve in Fig. 2. Here, as in Situation 1, the following cases can be realized when $\tau \in E^+(\tau_0)$:

a) the last zero $t_p(\tau_0) = t_*$ of the initial function $\Delta_{\tau_0}(\cdot)$ moves to the left forming a new switching point $t_p(\tau) = t_*$ of the control $u_\tau^0(\cdot)$ (see Subcase $\tilde{\text{A}}$ of Case B);
b) a new switching function $\bar{\Delta}_{\tau_0}(\cdot) = \bar{\Delta}_{\tau_0+0}(\cdot)$ is constructed which has the form shown by the thin curve in Fig. 1 (a sliding mode then arises (see Case C) or the first zero $t_1(\tau_0) = \tau_0$ of the new function $\bar{\Delta}_{\tau_0}(\cdot)$ moves to the right forming a new switching point $t_1(\tau) > \tau$ of the control $u_\tau^0(\cdot)$ (see Case A)), or the switching function has the form shown by the dashed curve in Fig. 3 (the multiple zero of the new switching function $\bar{\Delta}_{\tau_0}(\cdot)$ then generates two new switching points of the control $u_\tau^0(\cdot)$ (see Subcase $\tilde{\text{Б}}$ of Case B)).

The cases which have been enumerated are identified and the control $u^*(\tau)$, $\tau \in E^+(\tau_0)$ in each of them are constructed in the same way as in Situation 1.

Situation 3. This situation is investigated by analogy with Situations 1 and 2 only now all of the constructions start out from the switching function represented by the dashed curve in Fig. 3.

The rules for constructing the feedback in the neighbourhood of an irregular point τ_0 under the assumption that $\beta(\tau_0)=1$ and $n=p(\tau_0)+1$ have been described above.

The case when $\beta(\tau_0)=1$ and $n<p(\tau_0)+1$ is simpler to investigate since the Lagrange function $\psi_*(\tau)$ at the point τ_0 will always be continuous under such conditions. This case has also been investigated by us. However, it is not presented here for lack of space.

The case when $\beta(\tau_0)>1$ is more complex; it was investigated for linear systems earlier.[13,15]

Remark 2. In order to construct the feedback at regular points τ using the proposed rules, only a knowledge of the current state $z(\tau)$ is required while, at irregular points τ, the further minimal additional information $w(\tau+0)$ is required.

Remark 3. The investigations are carried out in a similar manner in the case of vector control.

4. Example

We will now illustrate the theoretical results obtained by a numerical example. We consider a model of an electric circuit,[4] the dynamics of which are described by the system of differential equations

$$\dot{z}(t) = \left\| \begin{matrix} z_2(t) \\ -z_1(t) + z_2(t)(1.4 - 0.14 z_2(t)^2) + 4u(t) \end{matrix} \right\| + w(t), \quad t \in T = [0, t_*] \tag{4.1}$$

where $w(t)$, $t \in T$ is a certain unknown perturbation, which acts on the system when it is functioning, $t_* = 4$. It is required to transfer system (4.1) from an initial state $z(0) = (-5, -5)^T$ into a specified final state $z(t_*) = x^* = (0.253, 1.610)^T$. The quality of the control is estimated by the functional (1.2).

On the basis of the results obtained, for the control of the real system (4.1) we shall use the solutions of the family of auxiliary problems which, in this example, has the form

$$OC(\tau): \begin{cases} \rho \to \min \\ \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -x_1(t) + x_2(t)(1.4 - 0.14 x_2(t)^2) + 4u(t) \\ x(\tau) = z(\tau), \quad x(t_*) = x_* \\ |u(t)| \le \rho, \quad \rho \ge 0, \quad t \in [\tau; t_*] \end{cases}$$

where $z(\tau)$ is the state of the real system (4.1) at the instant τ.

The instant $\tau=0$ is a regular point. The optimal control of the problem $OC(0)$ and the corresponding Lagrange vector have the form (a zero subscript henceforth indicates that the value of a function is taken at the point $\tau=0$)

$$u^0_{\tau=0}(t) = \rho_0, \quad t \in [0, t_{10}) \cup [t_{20}, t_*], \quad u^0_{\tau=0}(t) = -\rho_0, \quad t \in [t_{10}, t_{20})$$

where

$$\rho_0 = 1, \quad t_{10} = 1.121, \quad t_{20} = 3.31, \quad \psi_{*0} = (0.735, -0.403)^T$$

The structure and the vector of the defining elements are given by the relations

$$S_0 = \{p_0 = 2, k_0 = 1, l_{*0} = 0, l_0^* = 0, L_0 = \varnothing\}, \quad \Theta_0 = (t_{10}, t_{20}; \rho_0; \psi_{*0})$$

Time is considered as a continuous quantity in theoretical investigations. However, it is obvious that discretization is necessary in the case of a numerical application. We therefore subdivide the interval T into $r=t_*/h$ intervals with a certain fairly small step size $h>0$ and assume that the current state $z(ih)$ of system (4.1) can be measured at real instant of time of the form $\tau=ih$ ($i=1, \ldots, r$). On the basis of the results obtained above, we shall use the following "discrete" version of the feedback construction. For $i = 1, \ldots, i_1$ in the intervals $[ih, (i+1)h)]$, we shall construct the control according to the rule

$$u^*(t) = k\rho(ih), \quad t \in [ih, (i+1)h]$$

where $\Delta(ih)$ is the component of the vector

$$\Theta(ih) = (t_1(ih), t_2(ih); \rho(ih); \psi_*(ih))$$

which is the solution of system (2.16) when

$$p := p_0 = 2, \quad k := k_0 = 1, \quad \tau = ih$$

Here $i_1 > 1$ is a number which is such that the instant $\tau = i_1 h$ is the first irregular point. The actual value of the number i_1 is determined during the course of the calculations from an analysis of the current value of the vector $\Theta(i_1 h)$. For example, the point $i_1 h$ is assumed to be irregular if $t_1(i_1 h) \approx i_1 h$ or $t_1(i_1 h) \approx t_2(i_1 h)$ or $t_2(i_1 h) \approx t_*$.

In order to find the vector $\Theta(ih)$, we solve the corresponding system (2.16) using the known vector $\Theta((1-1)h)$ as the initial approximation.

The new structure of the solution $S((i_1+1)h)$ is determined at the instant of time $\tau = i_1 h$ when the regularity condition is violated, that is, the new parameters $p = p((i_1+1)h)$ and $k = k((i_1+1)h)$. The new vector $\Theta((i_1+1)h)$ and the corresponding system of non-linear equations (2.16) are formed using the new p and k. The control of system (4.1) is continued with this information.

In the example considered, $h = 0.05$ and the following was considered as the perturbation acting on the system

$$w(t) = \begin{cases} (\cos t, 4\sin t)^T, & t \in [0, 1] \\ (3, 3)^T, & t \in (1, 1.6] \\ -(0.2, 0.2)^T, & t \in (1.6, 2.8] \\ -(5.5)^T, & t \in (2.8, 3.1] \\ (0.2, 0.2)^T, & t \in (3.1, 4] \end{cases}$$

It was assumed that this perturbation was unknown in advance and only the current state $z(\tau)$ of system (4.1), generated by this perturbation and the control constructed up to the instant τ, was used for the calculations at the current instant τ.

Under the action of the perturbation $w(t)$ and the control calculated using the proposed rules, system (4.1) at the final instant of time was in the position

$$z(t_*) = (0.26, 1.602)^T$$

A graph of the control $u^*(t)$, $t \in T$ is shown in Fig. 4.

Note that, in the control of the dynamical system (4.1) which is being considered, the regularity conditions are violated three times at the instants $\tau^{(1)} = 0.95$, $\tau^{(2)} = 2.6$, $\tau^{(3)} = 3.65$ and that Situation 1 holds on each occasion. The coefficient γ_1 (see (3.6)) was calculated for each instant of irregularity and the new structure of the solution was constructed, the form of this being determined by the value of this coefficient. In the time interval $[0, \tau^{(1)}]$, the structure of the solution is identical to the initial structure and has the form

$$S_1(t) = S_0 = \{p = 2, k = 1, l_* = 0, l^* = 0, L = \varnothing\}, \quad t \in [0, \tau^{(1)}]$$

At the first instant of irregularity $\tau^{(1)} = 0.95$, the coefficient γ_1 takes a negative value. Since $n < p(\tau^{(1)}) + 1$, the parameter $p = p(\tau^{(1)})$ decreases by unity, the parameter $k = k(\tau^{(1)})$ changes sign and the new structure of the solution is given by the relations

$$S_2(t) = S(\tau^{(1)} + 0) = \{p = 1, k = -1, l_* = 0, l^* = 0, L = \varnothing\}, \quad t \in (\tau^{(1)}, \tau^{(2)}] \tag{4.2}$$

At the second instant of irregularity $\tau(2) = 2.6$, the value of the coefficient γ_1 is also negative. However, since $n = p(\tau^{(2)}) + 1$ now, the Lagrange function loses its discontinuity in this case and the new switching function vanishes

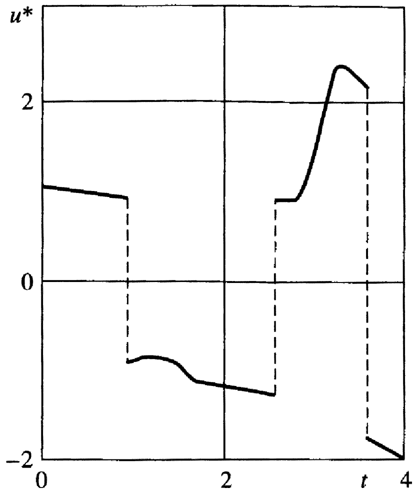

Fig. 4.

when $t = t_*$ (see Subcase Ã of Case B). The new structure of the solution $S_3(t) = S(\tau^{(2)} + 0)$, $t \in (\tau \in (\tau^{(2)}, \tau^{(3)}]$ differs from the structure of (4.2) by the replacement of $k = -1$ by $k = 1$.

At the third instant of irregularity $\tau^{(3)} = 3.65$, we have $n = p(\tau^{(3)}) + 1$, and the value of γ_1 falls in the interval (0, 1), which means that a sliding mode (see Case C) arises and the corresponding structure $S_4(t) = S(\tau^{(3)}) + 0)$, $t \in (\tau \in (\tau^{(3)}$, $t_*]$ differs from the structure of (4.2) in the replacement of $l_* = 0$ by $l_* = 1$.

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